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2006 J. Phys. A: Math. Gen. 39 7099

(http://iopscience.iop.org/0305-4470/39/22/020)

INSTITUTE OF PHYSICS PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND GENERAL

J. Phys. A: Math. Gen. **39** (2006) 7099–7104 doi:10.1088/0305-4470/39/22/020

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Received 21 December 2005, in final form 15 March 2006
Published 16 May 2006
Online at stacks.iop.org/JPhysA/39/7099

Abstract
Dirac's operator and Maxwell's equations in vacuum are derived in the algebra of split octonions. The approximations which lead to classical Maxwell–Heaviside equations from full octonionic equations are given. The non-existence of magnetic monopoles in classical electrodynamics is connected with the use of the associativity limit.

PACS numbers: 03.50.De, 02.10.De, 03.30.+p

1. Introduction

Maxwell's equations, which harbour many beautiful mathematical concepts, have been expressed in many forms since their discovery in 1873. Maxwell himself in his main book used the coordinate calculus [1]; however in the second edition he also included the quaternionic representation. The original equations were a system of 16 equations, quaternionic and the familiar vector forms consist of four equations, and the application of bi-quaternions (or Clifford algebras) results in a version of just one equation [2]. The octonionic form of Maxwell's equation is still absent in the literature. It was already mentioned that the vector algebra and Maxwell's equations should have connections with octonions [3]. In this paper, we want to show that in some approximation classical Maxwell–Heaviside equations can be written as the single continuity equation in the algebra of split octonions over the reals.

Octonions form the widest normed algebra after the algebras of real numbers, complex numbers and quaternions [4]. Since their discovery, almost three decades before Maxwell's equations, there have been various attempts to find appropriate uses for octonions in physics (see reviews [5]). One can point to the possible impact of octonions on colour symmetry [6], GUTs [7], representation of Clifford algebras [8], quantum mechanics [9], spacetime symmetries [10], field theory [11], formulations of wave equations [12], quantum Hall effect [13], strings and M-theory [14], etc.

In our previous papers [15], the model where the geometry of real world was described by the split octonions was introduced. In [16], the octonionic version of Dirac's equation was formulated. In this paper, except the derivation of octonionic Maxwell's equations in vacuum, we want to show that the symbolic form of Dirac's equations is just the result of the invariance of the intervals in the octonionic geometry.

2. Octonionic geometry

In [15], real physical signals were associated with the elements of split octonions,

$$s = ct + J_n x^n + j_n \hbar \lambda^n + I c \hbar \omega \qquad (n = 1, 2, 3), \tag{1}$$

where summing by the repeated indices is performed. In (1), the scalar unit is denoted as 1, the three vector-like objects as J_n, the three pseudo-vectors as j_n and the pseudo-scalar as I. The eight real parameters that multiply the basis units denote the time t, the special coordinates x^n and some quantities λ^n and ω with the dimensions of momentum^{-1} and energy^{-1}, respectively. The line element (1) also contains two fundamental constants of physics—the velocity of light c and Planck's constant $\hbar$. The appearance of these constants was connected with the existence of two different classes of zero divisors in the algebra of split octonions [15].

The algebra of the basis elements of split octonions can be written in the form

$$\begin{aligned} &J_n^2 = -j_n^2 = I^2 = 1, \qquad J_n j_m = -j_m J_n = -\epsilon_{nmk} J^k, \\ &J_n J_m = -J_m J_n = j_n j_m = -j_m j_n = \epsilon_{nmk} j^k, \\ &J_n I = -I J_n = j_n, \qquad j_n I = -I j_n = J_n, \end{aligned} \tag{2}$$

where ϵ_{nmk} is the fully antisymmetric tensor and $n, m, k = 1, 2, 3$. From these formulae, it is clear that to generate a complete eight-dimensional basis of split octonions the multiplication and distribution laws of only the three vector-like elements J_n are enough. The other two units j_n and I can be expressed as binary and triple products

$$j_n = \tfrac{1}{2}\epsilon_{nmk} J^m J^k, \qquad I = J_n j_n \tag{3}$$

(there is no summing in the second formula).

Using the conjugation rules of octonionic basis units

$$1^* = 1, \qquad J_n^* = -J_n, \qquad j_n^* = -j_n, \qquad I^* = -I, \tag{4}$$

one can find that the norm of (1) (interval)

$$s^2 = ss^* = c^2 t^2 - x_n x^n + \hbar^2 \lambda_n \lambda^n - c^2 \hbar^2 \omega^2 \tag{5}$$

has a (4 + 4)-signature and in general is not positively defined. However, as in the standard relativity we require

$$s^2 \geqslant 0. \tag{6}$$

In the classical limit $\hbar \to 0$, expression (5) reduces to the ordinary four-dimensional formula for spacetime intervals.

Differentiating (1) by the proper time τ, the proper velocity of a particle can be obtained:

$$\frac{\mathrm{d}s}{\mathrm{d}\tau} = \frac{\mathrm{d}t}{\mathrm{d}\tau}\left[c\left(1 + I\hbar\frac{\mathrm{d}\omega}{\mathrm{d}t}\right) + J_n\left(\frac{\mathrm{d}x^n}{\mathrm{d}t} + I\hbar\frac{\mathrm{d}\lambda^n}{\mathrm{d}t}\right)\right]. \tag{7}$$

Then the invariance of the norm (5) gives the relation

$$\beta = \frac{\mathrm{d}\tau}{\mathrm{d}t} = \sqrt{\left[1 - \hbar^2\left(\frac{\mathrm{d}\omega}{\mathrm{d}t}\right)^2\right] - \frac{v^2}{c^2}\left[1 - \hbar^2\left(\frac{\mathrm{d}\lambda^n}{\mathrm{d}x^n}\right)^2\right]}, \tag{8}$$

where

$$v_n = \frac{\mathrm{d}x_n}{\mathrm{d}t} \tag{9}$$

is the 3-velocity. So the modified Lorentz factor (8) contains extra terms and the dispersion relation in the (4 + 4)-space (5) has a form similar to that of double-special relativity models [17].

3. Symbolic form of Dirac's Equation

In [16], the octonionic form of Dirac's equation, which in some limit is equivalent to the standard one, was obtained. Here we want to demonstrate a simple derivation of Dirac's operator from the condition of invariance of the octonionic interval (5).

For the observers with the time parameters τ and t, we can write the relation

$$\mathrm{d}s = \pm c\,\mathrm{d}\tau = \pm c\,\mathrm{d}t\,\beta, \tag{10}$$

where β is expressed by (8). Dividing this relation by $\mathrm{d}\tau$ and multiplying it by the particle mass m, we find

$$\frac{1}{\beta}\left[mc\left(1 + I\hbar\frac{\mathrm{d}\omega}{\mathrm{d}t}\right) + J_n m\left(v^n + I\hbar\frac{\mathrm{d}\lambda^n}{\mathrm{d}t}\right)\right] = \pm cm. \tag{11}$$

Let us assume that

$$\frac{mc\hbar}{\beta}\frac{\mathrm{d}\omega}{\mathrm{d}t} = -\frac{e}{c}\varphi, \qquad \frac{m\hbar}{\beta}\frac{\mathrm{d}\lambda^n}{\mathrm{d}t} = -\frac{e}{c}A^n, \tag{12}$$

where φ and A^n are components of the electro-magnetic 4-potential. By this assumption, (5) takes a form similar to intervals in Finsler-type theories with field-dependent metrics. For the reviews on Finsler theories, see for example [18] and references therein.

Using the assumption (12), equation (11) takes the form

$$\left(\frac{\varepsilon}{c} - I\frac{e}{c}\varphi\right) + J_n\left(p^n - I\frac{e}{c}A^n\right) \mp mc = 0, \tag{13}$$

where $\varepsilon = mc^2/\beta$ and $p^n = mv^n/\beta$ are energy and momentum of the particle, respectively. This equation represents one of the zero divisors in the algebra of split octonions. The importance of zero divisors in physical applications of split algebras was specially noted in [19].

Equation (13), which we receive from the invariance of the interval (10), is the symbolic form of the four-dimensional Dirac's equation. The role of four γ-matrices here is played by the unit element of split octonions 1 and the three vector-like elements J_n. Instead of the ordinary complex unit i in (13), the basis element I is used, and the factor β transforms to the ordinary Lorentz formula if we use the limit $\hbar \to 0$ in definition (8).

4. Maxwell's equations in vacuum

The octonion that contains the electromagnetic potentials φ and A_n can be written as

$$A = -\varphi + J_n A^n + j_n B^n + Ib \qquad (n = 1, 2, 3), \tag{14}$$

where B^n and b correspond to the extra degrees of freedom in the octonionic algebra. Here we do not specify their meaning; we only want to obtain the approximations leading us to the classical Maxwell–Heaviside equations that give successful explanation of most experiments at low energies. Examples of problems in classical electrodynamics where the fields B^n and b can play a role are magnetic monopoles [20], longitudinal electrodynamic force [21], the Abraham–Minkowski controversy [22], etc.

To obtain the weak-field approximation in octonionic equations, let us mention that, since we require positivity of norms, the elements of split octonions should have a hierarchical structure. This means that the absolute value of the scalar element should be greater than other elements and so on. From (3) it is also clear that the pseudo-vector and pseudo-scalar units are secondary since they are expressed by the fundamental vector-like elements. The appearance of Planck's constant in the last two terms of (5) is another indication that in the classical limit we can neglect the values of pseudo-vector and pseudo-scalar components. So it is natural to consider that in (14)

$$|b|, |B^n| \ll |\varphi|, |A^n|, \tag{15}$$

and these components can be neglected. The invariance of octonionic intervals then will guarantee that this inequality would be preserved for different observers.

The octonionic differential operator can be written as

$$\nabla = \frac{1}{c}\left(\frac{\partial}{\partial t} + I\frac{1}{\hbar}\frac{\partial}{\partial \omega}\right) + J^n\left(\frac{\partial}{\partial x^n} + I\frac{1}{\hbar}\frac{\partial}{\partial \lambda^n}\right). \tag{16}$$

Here we can also assume that the influence of ω and λ^n can be ignored in the classical limit. The norm of ∇ when fields do not depend on ω and λ^n is the ordinary 4-d'Alembertian.

Assuming in (14) that B^n and b are small (or are constants) and A^n and φ are independent of ω and λ, we can define the electro-magnetic field in the form

$$\nabla A = F = \left(-\frac{1}{c}\frac{\partial \varphi}{\partial t} + \frac{\partial A^n}{\partial x^n}\right) + J_n E^n + j_n H^n, \qquad (n = 1, 2, 3), \tag{17}$$

where

$$E^n = \frac{1}{c}\frac{\partial A^n}{\partial t} - \frac{\partial \varphi}{\partial x_n}, \qquad H^n = \epsilon^{nmk}\frac{\partial A_k}{\partial x^m} \tag{18}$$

are components of 3-vectors of electric and magnetic fields, respectively.

Then we can postulate the Lorenz gauge (derived by L Lorenz in 1867, and not by H A Lorentz, as refereed in some modern papers)

$$\frac{1}{c}\frac{\partial \varphi}{\partial t} - \frac{\partial A^n}{\partial x^n} = 0, \tag{19}$$

or the weaker condition where zero in (19) is replaced by a constant, and write the continuity equation as the product of the octonions (16) and (17),

$$\nabla F = \frac{\partial E^n}{\partial x^n} + J_k\left(\frac{1}{c}\frac{\partial E^k}{\partial t} - \epsilon^{nmk}\frac{\partial H_m}{\partial x^n}\right) + j_k\left(\frac{1}{c}\frac{\partial H^k}{\partial t} + \epsilon^{nmk}\frac{\partial E_m}{\partial x^n}\right) + I\frac{\partial H^n}{\partial x^n} = 0. \tag{20}$$

Different signs in the second and third terms of this equation are the result of the use of the algebra (2), in particular

$$J_n j_m = -\epsilon_{nmk} J^k, \qquad J_n J_m = \epsilon_{nmk} j^k. \tag{21}$$

Equating to zero coefficients in front of the four octonionic basis units in (20) results in the full set of the homogeneous Maxwell's equations.

We can also write the octonionic current function in the form

$$\varrho = \rho + J_n \frac{1}{c}\sigma^n, \tag{22}$$

where ρ is the electric charge density and σ^n are the components of the electric current vector. As before, we ignored pseudo-vector and pseudo-scalar parts in (22).

Finally, we can write the complete set of inhomogeneous Maxwell's equations as one single octonionic equation

$$\nabla F = \varrho. \tag{23}$$

As it is clear from (20) and (22) in (23), as in standard electrodynamics, the magnetic current is absent. This is the result of ignoring pseudo-vector and pseudo-scalar terms in (22), and of re-appearance of these kinds of terms in (20) via octonionic products. Non-associativity of octonions is mainly governed by j_n and I. So the non-existence of magnetic monopoles in classical electrodynamics can be explained as the use of associativity limit.

5. Conclusion and discussion

In this paper, simple octonionic forms of Dirac's operator and Maxwell's equations in vacuum were derived. In the classical limit, there is no indication of non-associativity for the electromagnetic field, and the derived octonionic Maxwell's equation (23) is similar to the bi-quaternion formulations [2]. However, split octonions that incorporate three vector-like elements should give a more successful generalization of classical electrodynamics since non-associativity (which distinguishes octonions from other normed algebras) also exists in the algebra of Euclidean 3-vectors used in the classical Maxwell–Heaviside equations. The only new feature of the octonionic formalism in the approximation used in this paper is the observation that the non-existence of magnetic monopoles in classical electrodynamics is connected with ignoring non-associativity. In the case of strong fields, the pseudo-vector and pseudo-scalar parts of octonions cannot be neglected, equations will become more complicated and we expect to find new effects in future papers.

Acknowledgments

The author would like to acknowledge the hospitality extended during his visits at the Abdus Salam International Centre for Theoretical Physics where this work was done.

References

[1] Maxwell J C 1954 *Treatise on Electricity and Magnetism* (New York: Dover)

[2] Hestenes D 1966 *Space-Time Algebra* (New York: Gordon and Breach)
Negi O P S, Bisht S and Bisht P S 1998 *Nuovo Cimento* B **113** 1449
Baylis W E 1999 *Electrodynamics: A Modern Geometric Approach* (Boston, MA: Birkhäuser)
Grudsky S M, Khmelnytskaya K V and Kravchenko V V 2004 *J. Phys.: Condens. Matter* **37** 4641 (*Preprint* math-ph/0309062)

[3] Silagadze Z K 2002 *Ann. Fond. Louis Broglie* **27** 241 (*Preprint* hep-ph/0106235)

[4] Schafer R 1995 *Introduction to Non-Associative Algebras* (New York: Dover)
Springer T A and Veldkamp F D 2000 *Octonions, Jordan Algebras and Exceptional Groups Springer Monographs in Mathematics* (Berlin: Springer)
Baez J 2002 *Bull. Amer. Math. Soc.* **39** 145 (*Preprint* math.RA/0105155)

[5] Finkelstein F 1996 *Quantum Relativity: A Synthesis of the Ideas of Einstein and Heisenberg* (Berlin: Springer)
Emch G 1972 *Algebraic Methods in Statistical Mechanics and Quantum Field Theory* (New York: Wiley)
Gürsey F and Tze C 1996 *On the Role of Division, Jordan and Related Algebras in Particle Physics* (Singapore: World Scientific)
Lõhmus J, Paal E and Sorgsepp L 1998 *Acta Appl. Math.* **50** 3

[6] Günaydin M and Gürsey F 1973 *Lett. Nuovo Cimento* **6** 401
Günaydin M and Gürsey F 1973 *J. Math. Phys.* **14** 1651
Günaydin M and Gürsey F 1974 *Phys. Rev.* D **9** 3387
Adler S L 1980 *Phys. Rev.* D **21** 2903
Morita K 1981 *Prog. Theor. Phys.* **65** 787

[7] Kugo T and Townsend P 1983 *Nucl. Phys.* B **221** 357
Sudbery A 1984 *J. Phys. A: Math. Gen.* **12** 939
Dixon G 1990 *Nuovo Cimento* B **105** 349

Howe P S and West P C 1992 *Int. J. Mod. Phys.* A **7** 6639
De Leo S 1996 *Int. J. Theor. Phys.* **35** 1821 (*Preprint* hep-th/9506008)
[8] Porteous I R 1995 *Clifford Algebras and the Classical Groups* (Cambridge: Cambridge University Press)
Okubo S 1995 *Introduction to Octonions and Other Non-Associative Algebras in Physics* (Cambridge: Cambridge University Press)
Lounesto P 2001 *Clifford Algebras and Spinors* (Cambridge: Cambridge University Press)
Carrion H L, Rojas M and Toppan F 2003 *J. High Energy Phys.* JHEP04(2003)040 (*Preprint* hep-th/0302113)
[9] Finkelstein D, Jauch J M, Schiminovich S and Speiser D 1962 *J. Math. Phys.* **3** 207
Finkelstein D, Jauch J M, Schiminovich S and Speiser D 1963 *J. Math. Phys.* **4** 788
Emch G G 1963 *Helv. Phys. Acta* **36** 739
Horwitz L P and Biedenharn L C 1965 *Helv. Phys. Acta* **38** 385
Horwitz L P and Biedenharn L C 1984 *Ann. Phys.* **157** 432
Günaydin M, Piron C and Ruegg H 1978 *Commun. Math. Phys.* **61** 69
De Leo S and Rotelli P 1994 *Prog. Theor. Phys.* **92** 917 (*Preprint* hep-th/9401009)
Dzhunushaliev V 2003 *Found. Phys. Lett.* **16** 57 (*Preprint* hep-th/0103172)
Dzhunushaliev V 2003 *Found. Phys. Lett.* **16** 265 (*Preprint* hep-th/0208088)
Dzhunushaliev V 2005 *Preprint* hep-th/0502216
Dzhunushaliev V 2005 *Preprint* hep-th/0507032
[10] Gürsey F 1987 *Symmetries in Physics (1600–1980): Proc. 1st International Meeting on the History of Scientific Ideas* (Barcelona: Bellaterra)
De Leo S 1996 *J. Math. Phys.* **37** 2955 (*Preprint* hep-th/9508011)
[11] Morita K 1982 *Prog. Theor. Phys.* **67** 1860
Nash C and Joshi G C 1987 *J. Math. Phys.* **28** 463
Nash C and Joshi G C 1988 *Int. J. Theor. Phys.* **27** 409
Adler S L 1995 *Quaternion Quantum Mechanics and Quantum Field* (New York: Oxford University Press)
[12] Kurdgelaidze D 1985 *Acta Phys. Hung.* **57** 79
Davies A J and Joshi G C 1986 *J. Math. Phys.* **27** 3036
Marques-Bonham S 1988 *J. Math. Phys.* **29** 2127
Marques-Bonham S 1990 *J. Math. Phys.* **31** 1478
Marques-Bonham S 1991 *J. Math. Phys.* **32** 1383
De Leo S and Rotelli P 1996 *Mod. Phys. Lett.* A **11** 357 (*Preprint* hep-th/9509059)
De Leo S and Rotelli P 1998 *Int. J. Theor. Phys.* **37** 1511 (*Preprint* hep-ph/9605255)
De Leo S and Abdel-Khalek K 1996 *Prog. Theor. Phys.* **96** 823 (*Preprint* hep-th/9609032)
De Leo S and Abdel-Khalek K 1996 *Prog. Theor. Phys.* **96** 833 (*Preprint* hep-th/9609033)
De Leo S and Abdel-Khalek K 1998 *Int. J. Theor. Phys.* **37** 1945 (*Preprint* hep-th/9905123)
Toppan F 2005 *AIP Conf. Proc.* **767** 316 (*Preprint* hep-th/0503210)
[13] Bernevig B A, Hu J-P, Toumbas N and Zhang S-C 2003 *Phys. Rev. Lett.* **91** 236803 (*Preprint* cond-mat/0306045)
[14] Fairlie D B and Manogue A C 1986 *Phys. Rev.* D **34** 1832
Chung K W and Sudbery A 1987 *Phys. Lett.* B **198** 161
Lukierski J and Toppan F 2002 *Phys. Lett.* B **539** 266 (*Preprint* hep-th/0203149)
Lukierski J and Toppan F 2003 *Phys. Lett.* B **567** 125 (*Preprint* hep-th/0212201)
Lukierski J and Toppan F 2004 *Phys. Lett.* B **584** 315 (*Preprint* hep-th/0312083)
[15] Gogberashvili M 2002 *Preprint* hep-th/0212251
Gogberashvili M 2005 *Adv. Appl. Clifford Algebras* **15** 55 (*Preprint* hep-th/0409173)
[16] Gogberashvili M 2006 *Int. J. Mod. Phys.* A **21** (*Preprint* hep-th/0505101)
[17] Amelino-Camelia G 2002 *Int. J. Mod. Phys.* D **11** 35 (*Preprint* gr-qc/0012051)
Mabuejo J and Smolin L 2002 *Phys. Rev. Lett.* **88** 190403 (*Preprint* hep-th/0112090)
[18] Asanov G S 1985 *Finsler Geometry, Relativity and Gauge Theories* (Dordrecht: Reidel)
Vacaru S, Stavrinos P, Gaburov E and Gonta D 2006 Clifford and Riemann-Finsler structure in geometric mechanics and gravity (Bucharest: Geometry Balkan Press) *Preprint* gr-qc/0508023
[19] Sommerfeld A 1953 *Atombau und Spektrallinien, II Band* (Braunschweig: Vieweg)
[20] Giacomelli G *et al* 2000 Magnetic monopole bibliography *Preprint* hep-ex/0005041
[21] Graneau A 1985 *Ampere–Neumann Electrodynamics of Metals* (Nonantum, MA: Hadronic)
Johansson L 1996 *Longitudinal Electrodynamic Forces and Their Possible Technological Applications* (available at http://www.df.lth.se/snorkelf/LongitudinalMSc.pdf)
Chubykalo A E and Smirnov-Rueda R 1997 *Mod. Phys. Lett.* A **12** 1 (*Preprint* hep-th/9608038)
[22] Penfield P and Haus H A 1967 *Electrodynamics of Moving Media* (Cambridge, MA: MIT Press)
Robinson F N H 1975 *Phys. Rep.* **16** 313